Functions of a random variable

When deriving the probability distribution of Y, where Y is a function of X, $Y = u(X)$ say, from the known distribution of X, the procedure is straightforward for discrete variables but for continuous variables there are several methods available. Methods are illustrated in this section based first on the distribution function $F_X(x)$, and second directly on the PDF $f_X(x)$. Often both methods can be used, although one may be easier and quicker to carry out than the other.

F and f will denote the CDF and PF/PDF respectively of Y; F_X and f_X will denote the CDF and PF/PDF respectively of X.

Discrete random variables

All that has to be done is to find the value y of Y which corresponds to the value (or values) x of X – the probabilities in the given distribution of X are unaffected.

For example suppose that X has the following distribution:

X	0	1	2
$P(X=x)$	0.3	0.5	0.2

Let $Y = 2X + 1$. Then it will have the following distribution:

Y	1	3	5
$P(Y=y)$	0.3	0.5	0.2

Question 1

The random variable Z has probability function:

$$P(Z=z) = \binom{n}{z} \mu^z (1-\mu)^{n-z} \quad z = 0, 1, \ldots, n$$

where n is a positive integer and $0 < \mu < 1$.

Determine the probability function of $Y = Z/n$.

Continuous random variables

$F(Y) = P(Y \leq y)$ $[= P(Y < y)]$ is required. The event '$Y < y$' is equivalent to the event '$u(X) < y$' (as $Y = u(X)$), so this latter event is expressed in terms of the values which X must take. Then $P[u(X) < y]$ can be found from F_X. We rearrange the formula to get $P[X < u^{-1}(y)]$ and then since $F_X(x) = P[X < x]$, we just need to find $F_X(u^{-1}(y))$. This can be done either by using the F_X formula from the *Tables* or by integration.

Having found $F(y)$, obtain the PDF $f(y)$, if required, by differentiation. $F(y)$ and/or $f(y)$ may be recognisable as the distribution function/density function of a "standard" distribution, *eg* uniform, gamma, normal. These standard distributions can be found in in the *Tables*.

For example, if $y = u(x)$ is such that a unique inverse $x = w(y) = u^{-1}(y)$ exists and $u(x)$ is an increasing function, then:

$$F(y) = P(Y < y) = P[u(X) < y] = P[X < w(y)] = F_X[w(y)]$$

and hence differentiating using the chain rule $f(y) = f_X[w(y)]\dfrac{dw(y)}{dy}$.

In the case that $u(x)$ is decreasing:

$$F(y) = P(Y < y) = P[u(X) < y] = P[X > w(y)] = 1 - F_X[w(y)]$$

and $f(y) = -f_X[w(y)]\dfrac{dw(y)}{dy}$.

As $\dfrac{dw(y)}{dy}$ is negative in this case, both cases are summed up in the result:

$$f(y) = f_X[w(y)]\left|\frac{dw(y)}{dy}\right|$$

Example

The random variable T has an exponential distribution (see the *Tables*) with PDF:

$$f(t)=\lambda e^{-\lambda t}, \quad t>0$$

By considering the distribution function, determine the distribution of $U=e^{-\lambda T}$.

Solution

Consider the distribution function of U:

$$F_U(u)=P(U\le u)=P(e^{-\lambda T}\le u)=P(-\lambda T\le \ln u)=P\left(T\ge -\tfrac{\ln u}{\lambda}\right)$$

Expressing this as an integral:

$$\int_{-\frac{1}{\lambda}\log u}^{\infty} \lambda e^{-\lambda t}\,dt=\left[-e^{-\lambda t}\right]_{-\frac{1}{\lambda}\log u}^{\infty}=0-(-u)=u$$

Alternatively, we could use the distribution function of the exponential distribution in the Tables:

$$P\left(T\ge -\tfrac{\ln u}{\lambda}\right)=1-P\left(T\le -\tfrac{\ln u}{\lambda}\right)=1-F_T\left(-\tfrac{\ln u}{\lambda}\right)=1-\left(1-e^{-\lambda\times-\frac{\ln u}{\lambda}}\right)=u$$

Having determined $F_U(u)$, we need to differentiate to obtain $f_U(u)$:

$$f_U(u)=F_U'(u)=1$$

Since T can take values in the range $0<T<\infty$, U can take values in the range $0<U<1$. So U has a $U(0,1)$ distribution.

Alternatively, jumping straight to the formula $f_U(u)=f_T[w(u)]\times\left|\frac{d}{du}w(u)\right|$ *we obtain:*

$$u=e^{-\lambda t} \quad\Rightarrow\quad t=w(u)=-\tfrac{1}{\lambda}\ln u \quad\Rightarrow\quad \tfrac{d}{du}w(u)=-\tfrac{1}{\lambda u}$$

$$\Rightarrow\quad f_U(u)=\lambda e^{-\lambda\times-\frac{1}{\lambda}\ln u}\times\left|-\tfrac{1}{\lambda u}\right|=e^{\ln u}\times\tfrac{1}{u}=u\times\tfrac{1}{u}=1$$

Question 2

(i) Determine the cumulative distribution function for the random variable having the PDF:

$$f(x) = 2\beta x e^{-\beta x^2}, \qquad x > 0$$

where β is a positive constant.

(ii) Hence, derive the PDF of $Y = X^2$ if X has the distribution above.

The method used above relies on calculating probabilities using a distribution function or integration. However, some distributions have complicated PDFs so that the integral needed to calculate the distribution functions cannot be evaluated analytically. For example:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

For these an alternative method is required which is listed in the appendix to this chapter.

Solutions

Solution 1

The probability function of Y is given by:

$$P(Y=y)=P\left(\frac{Z}{n}=y\right)=P(Z=ny)$$

So using the PF of Z given in the question:

$$P(Y=y)=P(Z=ny)=\binom{n}{ny}\mu^{ny}(1-\mu)^{n-ny}$$

The original PF was for $z=0,1,\dots,n$ whereas now it is for $y=0,\frac{1}{n},\dots,1$.

Solution 2

The cumulative distribution function is:

$$F(x)=\int_0^x 2\beta te^{-\beta t^2}\,dt=\left[-e^{-\beta t^2}\right]_0^x=1-e^{-\beta x^2}$$

So the distribution function of Y is:

$$F_Y(y)=P(Y\le y)=P(X^2\le y)=P(-\sqrt{y}\le X\le\sqrt{y})=P(X\le\sqrt{y})$$

since X can only take positive values. This is equal to $F_X(\sqrt{y})=1-e^{-\beta y}$.

Therefore:

$$f_Y(y)=F_Y'(y)=\beta e^{-\beta y}$$